Solutions to H.W #3

1. $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix}$

$= 2(3-2\cdot 0) + (4-6) = 6-2 = 4$

2. $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix} = 1\begin{vmatrix} 9 & 16 \\ 16 & 25 \end{vmatrix} - 4\begin{vmatrix} 4 & 16 \\ 9 & 25 \end{vmatrix} + 9\begin{vmatrix} 4 & 9 \\ 9 & 16 \end{vmatrix} =$

$= -31 - 4(-44) + 9(-17) = -8$

3. $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ -16 & -12 & -8 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ -4(4 & 3 & 2) \end{vmatrix} = -4\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 0$

4. $\begin{vmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} = 3\cdot 2\cdot(-1)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$

$= 3\cdot 2\cdot(-1)\cdot 4 = -4!$

5. $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ -5a_{11} & -5a_{12} & -5a_{13} \end{vmatrix} = -5\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = -5(-1)\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= -5(-1)^2\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -5\cdot 4 = -20.$

6. $$\begin{vmatrix} a_{11}+2a_{21} & a_{12}+2a_{22} & a_{13}+2a_{23} \\ 155a_{11}+a_{21} & 155a_{12}+a_{22} & 155a_{13}+a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} =$$

$$= \begin{vmatrix} \vec{a}+2\vec{b} \\ 155\vec{a}+\vec{b} \\ -\vec{c} \end{vmatrix}$$ where $\vec{a} = (a_{11}, a_{12}, a_{13})$

$\vec{b} = (a_{21}, a_{22}, a_{23})$, and $\vec{c} = (a_{31}, a_{32}, a_{33})$,

thus $$\begin{vmatrix} \vec{a}+2\vec{b} \\ 155\vec{a}+\vec{b} \\ -\vec{c} \end{vmatrix} = \begin{vmatrix} \vec{a} \\ 155\vec{a} \\ -\vec{c} \end{vmatrix} + \begin{vmatrix} \vec{a} \\ \vec{b} \\ -\vec{c} \end{vmatrix} + \begin{vmatrix} 2\vec{b} \\ 155\vec{a} \\ -\vec{c} \end{vmatrix} + \begin{vmatrix} 2\vec{b} \\ \vec{b} \\ -\vec{c} \end{vmatrix} =$$

$$= \begin{vmatrix} \vec{a} \\ \vec{b} \\ -\vec{c} \end{vmatrix} + \begin{vmatrix} 2\vec{b} \\ 155\vec{a} \\ -c \end{vmatrix} = -\begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} + 2\cdot 155 \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} = -4 + 2\cdot 155\cdot 4 =$$

$$= 4(310-1) = 4\cdot(309) = 1236$$

7. $\vec{a} = (1,-2,1)$, $\vec{b} = (2,1,1)$

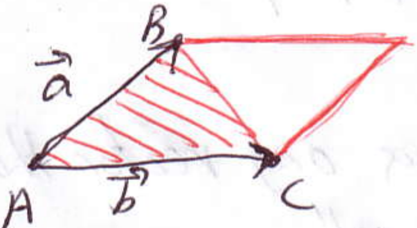
$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (-2\cdot 1-1)i - (1\cdot 1 - 1\cdot 2)j + (1+4)k$$

$$= (-3, 1, 5)$$

8. The area of that parallelogram is just

$\|\vec{a}\times\vec{b}\| = \sqrt{(-3)^2+1^2+5^2} = \sqrt{9+1+25} = \sqrt{35}$

9. lable the vertices By A, B, C where $A=(0,0,0)$



Let $\vec{a} = \vec{B-A} = (1,1,1)$ and $\vec{b} = \vec{C-A} = (0,-2,3)$

then the area of the triangle with sides $\vec{a}$ and $\vec{b}$ is half the area of the parallelogram spanned by $\vec{a}$ & $\vec{b}$.

Thus $\frac{1}{2}\|\vec{a}\times\vec{b}\| =$ Area of triangle.

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -2 & 3 \end{vmatrix} = (3+2)i - (3-0)j + (-2-0)k$$

$= (5,-3,-2)$ so

$\frac{1}{2}\|\vec{a}\times\vec{b}\| = \frac{1}{2}\sqrt{5^2+3^2+2^2} = \frac{1}{2}\sqrt{25+9+4} = \frac{1}{2}\sqrt{38}$.

10. $\vec{a} = (2,1,-1)$, $\vec{b} = (5,0,-3)$, $\vec{c} = (1,-2,1)$

Volume of paralelepiped is $\left|\det\begin{pmatrix} 2 & 1 & -1 \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}\right| =$

$= \left|\det\begin{pmatrix} 5 & 0 & -3 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}\right| = \left| 5\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \right| =$

$= |5(-1) - 3(-4-1)| = |-5+3\cdot 5| = |10| = 10$

11. $k, -k$ are the only unit vectors orthogonal to i, j.

12. $\vec{a} = (2, -4, 3)$ $\vec{b} = (-4, 8, -6)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -4 & 3 \\ -4 & 8 & -6 \end{vmatrix} = (0, 0, 0)$$

because $\vec{b} = -2\vec{a}$. Thus any vector that is orthogonal to $\vec{a}$ is orthogonal to $\vec{b}$ as well.

The set of vectors orthogonal to $\vec{a}$ is given by the equation $(2, -4, 3) \cdot (x, y, z) = 0 \Rightarrow$

$2x - 4y + 3z = 0$. The set of all vectors orthogonal to $(2, -4, 3)$ is therefore $\{(x, y, z) : 2x - 4y + 3z = 0\} =$

$= \{(x, y, \frac{-2x}{3} + \frac{4y}{3}) ; x, y \in \mathbb{R}\} = \{(3x, 3y, -2x + 4y) ; x, y \in \mathbb{R}\}$

Thus all normal vectors to $(2, -4, 3)$ of unit size are of the form $\dfrac{(3x, 3y, -2x + 4y)}{\sqrt{9x^2 + 9y^2 + (-2x + 4y)^2}}$.

13. $\vec{v}$ is the normal to the plane. Thus $\vec{v} \cdot (x-1, y-1, z-1) = 0$ is the desired equation:

$(1, 2, 3) \cdot (x-1, y-1, z-1) = 0 \Rightarrow x - 1 + 2(y-1) + 3(z-1) = 0$

$\Rightarrow x + 2y + 3z - 1 - 2 - 3 = 0 \Rightarrow x + 2y + 3z - 6 = 0$,

So our plane is the set $P = \{(x, y, z) : x + 2y + 3z - 6 = 0\}$

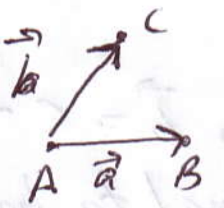
14. To be perpendicular to a line, the plane must be perpendicular to the direction of the line.
Since the line is given by $L(t) = (5,0,2)t + (3,-1,1)$, it follows that the directional vector of the line is $(5,0,2)$.
Hence the desired plane is determined by the equation

$$(5,0,2)\cdot(x-5, y+1, z-0) = 0 \Rightarrow$$

$$5(x-5) + 0(y+1) + 2z = 0 \Rightarrow$$

$$5(x-5) + 2z = 0.$$

15. 3 points determine a line. Let $A = (0,0,0)$, $B = (2,0,-1)$, $C = (0,4,3)$



This plane can be written parametrically as

$$P = \{(x,y,z) : (x,y,z) = s\vec{a} + t\vec{b}\} = \{(x,y,z) : (x,y,z) =$$

$$= s(2,0,-1) + t(0,4,3)\} = \{(2s, 4t, -s+3t) ; s,t \in (-\infty, \infty)\}$$

If a formula in x, y, z is desired, it can be obtained as follows: $(2s, 4t, -s+3t) = (x,y,z)$

$$\Rightarrow x = 2s \quad y = 4t \Rightarrow -s = -\frac{x}{2}, \; 3t = \frac{3y}{4}$$

Hence $z = -5+3t \Rightarrow -\frac{x}{2} + \frac{3}{4}y$.

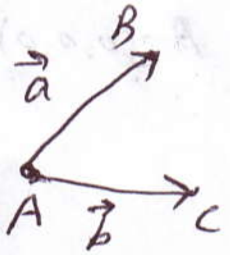
It follows that $P = \{(x, y, -\frac{x}{2} + \frac{3}{4}y) : x, y \in \mathbb{R}\}$

$= \{(x,y,z) : z = -\frac{x}{2} + \frac{3}{4}y\} =$

$= \{(x,y,z) : -2x - 3y + 4z = 0\}$

16. This is a similar problem to the one just solved. We will consequently solve it using a different method

let $A = (0,0,5)$, $B = (2,-1,3)$, $C = (5,7,-1)$



then $\vec{a} = B - A = (2,-1,-2)$, $\vec{b} = C - A = (5,7,-6)$.
The normal to the plane is normal to both $\vec{a}$ & $\vec{b}$.

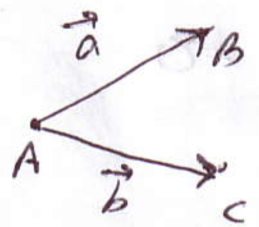
$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & -2 \\ 5 & 7 & -6 \end{vmatrix} = \left(6+14, -(-12+10), 14+5\right)$$

$= (20, 2, 29)$.

The plane must be orthogonal to $(20,2,19)$ and the point $A = (0,0,5)$ must be on the plane.
Thus the equation of the plane is $(20,2,19)\cdot(x,y,z-5) = 0 \Rightarrow 20x + 2y + 19(z-5) = 0$.

17. To contain the two parallel lines, the plane must contain the points $L(0)$, $L(1)$, and $S(0)$.

That is the points $A = L(0) = (0,1,-2)$, $B = L(1) =$
$= (0,1,-2)+(2,3,-1) = (2,4,-3)$, and $C = S(0) = (2,-1,0)$



$\vec{a} = B - A = (2,3,-1)$ $\quad \vec{b} = C - A = (2,-1,0) - (0,1,-2)$
$= (2,-2,2)$

$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 2 & -2 & 2 \end{vmatrix} = (6-2, -(4+2), -4-6)$$

$= (4,-6,-10)$

Hence the plane is given by the equation

$\vec{n}\cdot(x-0, y-1, z+2) = 0 \implies (4,-6,-10)\cdot(x, y-1, z+2) = 0$

$\implies 4x - 6(y-1) - 10(z+2) = 0.$

18. Let $P_1 = \{(x,y,z) : x+2y+z=0\}$ & $P_2 = \{(x,y,z) : x-3y-z=0\}$

then the intersection is given by $P_1 \cap P_2 =$

$= \{(x,y,z) : x-3y-z=0 \ \& \ x+2y+z=0\}$

This set is the solution to two equations in 3 unknowns

$$\begin{cases} x-3y-2=0 \\ x+2y+2=0 \end{cases}$$

This system implies that

$$+\begin{cases} x-3y-2=0 \\ x+2y+2=0 \end{cases} = 2x-y=0$$

or $y=2x$

replacing in the original equations, we see that

$$\begin{cases} x-3(2x)-2=0 \\ x+2(2x)+2=0 \end{cases} \Rightarrow \begin{cases} -5x-2=0 \\ 5x+2=0 \end{cases}$$

both equations imply that $z=-5x$

It follows that $P_1 \cap P_2 = \{(x,y,z) ; y=2x, z=-5x, x\in\mathbb{R}\}$
$= \{(x,2x,-5x) ; x\in\mathbb{R}\} = \{(t,2t,-5t) ; t\in\mathbb{R}\}$.

In other words, the intersection of the two planes is given by $L(t)=t(1,2,-5)$ which is a line spanned by the vector $(1,2,-5)$.

19. a) Two planes are parallel iff they have the same normal $\vec{n} = (A, B, C)$

Let (a_1, a_2, a_3) be a point on the plane P_1 and let (b_1, b_2, b_3) be a point on the plane P_2.

Then $P_1 = \{(x,y,z) : A(x-a_1)+B(y-a_2)+C(z-a_3) = 0\}$
and $P_2 = \{(x,y,z) : A(x-b_1)+B(y-b_2)+C(z-b_3) = 0\}$

Hence $P_1 = \{(x,y,z) : Ax+By+Cz = D_1\}$
and $P_2 = \{(x,y,z) : Ax+By+Cz = D_2\}$ where
$D_1 = a_1A + a_2B + a_3C$ and $D_2 = b_1A + b_2B + b_3C$.

Notice that $P_1 \cap P_2$ is the set of all points that satisfy the system of equations

$$\begin{cases} Ax+By+Cz = D_1 \\ Ax+By+Cz = D_2. \end{cases}$$

Subtracting one equation from the other, we see that $D_1 - D_2 = 0$ or $D_1 = D_2$.

Thus, two parallel planes intersect iff and only if they are in fact the same plane.

b) The intersection of two non parallel planes is a line* To see that this is always the case
(* or the empty set)

consider two arbitrary nonparallel planes

$P_1 = \{(x,y,z) : A_1 x + B_1 y + C_1 z + D_1 = 0\}$ &

$P_2 = \{(x,y,z) : A_2 x + B_2 y + C_2 z + D_2 = 0\}$

These planes are not parallel iff $\vec{n}_1 = (A_1, B_1, C_1)$ and $\vec{n}_2 = (A_2, B_2, C_2)$ are linearly independent.

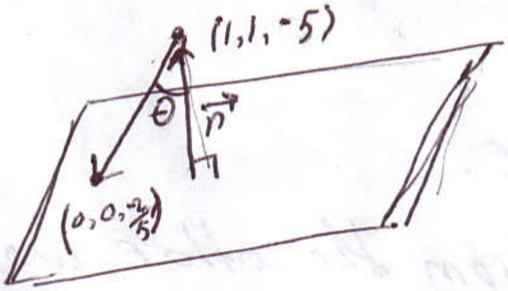
let $\vec{m} = \vec{n}_1 \times \vec{n}_2 \neq \vec{0}$

If $P_1 \cap P_2 \ni (\alpha, \beta, \lambda)$ then $(x-\alpha, y-\beta, z-\lambda)$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ for all $(x,y,z) \in P_1 \cap P_2$.

Thus, $(x-\alpha, y-\beta, z-\lambda)$ is parallel to $\vec{m}$.

In particular any point $(x,y,z) \in P_1 \cap P_2$, then $(x-\alpha, y-\beta, z-\lambda) = t\vec{m}$ for some $t \in \mathbb{R}$. Hence $(x,y,z) = (\alpha, \beta, \lambda) + t\vec{m}$.

20.



Let $\vec{n} = (12, 13, 5)$ then $\vec{n}$ is perpendicular to the plane, the distance from $(1,1,-5)$ to the plane is the size of the projection of $\vec{a} = (0,0,-\frac{2}{5}) - (1,1,-5) = (-1,-1,\frac{25-2}{5})$ $= (-1,-1,\frac{23}{5})$ on $\vec{n}$ $\|P_{\vec{n}}(-1,-1,\frac{23}{5})\| = \left\| \frac{(-1,-1,\frac{23}{5}) \cdot (12,13,5)}{\|(12,13,5)\|^2} (12,13,5) \right\|$

$= \left| \frac{-12-13+23}{\sqrt{12^2+13^2+5^2}} \right| = \frac{2}{\sqrt{338}}$

21. The equations $\vec{x}\times\vec{a}=\vec{b}$ and $\vec{x}\cdot\vec{a}=\|\vec{a}\|$ determines a unique vector $\vec{x}$ if $\vec{a}\neq\vec{0}$.

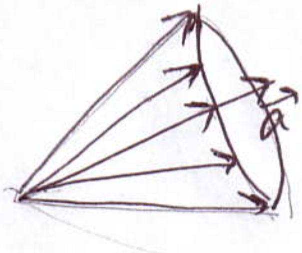
To see this, suppose $\vec{a}\neq\vec{0}$. Then $\vec{x}\cdot\vec{a}=\|\vec{x}\|\|\vec{a}\|\cos\theta=\|\vec{a}\|$
$\Rightarrow \|\vec{x}\|\cos\theta=1$

Notice $\|\vec{x}\times\vec{a}\|=\|\vec{x}\|\|\vec{a}\|\sin\theta=\|\vec{b}\| \Rightarrow \|\vec{x}\|\sin\theta=\frac{\|\vec{b}\|}{\|\vec{a}\|}$.

Hence $\|\vec{x}\|=\sqrt{\|\vec{x}\|^2}=\sqrt{(\|\vec{x}\|\cos\theta)^2+(\|\vec{x}\|\sin\theta)^2}=$
$=\sqrt{1+\frac{\|\vec{b}\|^2}{\|\vec{a}\|^2}}=\frac{\sqrt{\|\vec{a}\|^2+\|\vec{b}\|^2}}{\|\vec{a}\|}$. In particular,

the size of $\vec{x}$, $\|\vec{x}\|$ is determined. Also, since $\cos\theta=\frac{1}{\|\vec{x}\|}=$
$=\frac{1}{\left(\frac{\sqrt{\|\vec{a}\|^2+\|\vec{b}\|^2}}{\|\vec{a}\|}\right)}=\frac{\|\vec{a}\|}{\sqrt{\|\vec{a}\|^2+\|\vec{b}\|^2}}$, $\theta=\cos^{-1}\left(\frac{\|\vec{a}\|}{\sqrt{\|\vec{a}\|^2+\|\vec{b}\|^2}}\right)$.

Hence the angle between $\vec{x}$ and $\vec{a}$ is determined as well.
The set of all vectors of fixed size and at a fixed angle from a vector $\vec{a}$ forms a cone about $\vec{a}$:



because $\vec{b}$ is a fixed vector that is orthogonal to $\vec{x}$ and $\vec{a}$, it follows that there can be only one such $\vec{x}$.